Master Thesis Poster, On the Winding Numbers

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Abstract

In this thesis, we begin by introducing the concept of the winding number for plane curves, a fundamental idea that finds applications across various mathematical domains, including topology, complex integration, and combinatorics. We carefully define the winding number in each of these contexts and demonstrate the equivalence of these definitions. Our main theoretical contribution is presented in Chapter 5, where we develop a version of the De Rham Theorem for the plane, framed through the lens of winding numbers. This result forms the foundation for Chapter 6, which explores a range of applications in complex analysis and topology. A key strength of this work is its unified focus on winding numbers, with proofs across different theorems intentionally crafted to highlight their shared structure, showcasing the power of winding numbers as a versatile tool in resolving diverse mathematical problems.

One highlight: Artin's criterion

Let Ω be a domain, a cycle c in Ω is a boundary iff $\forall p \notin \Omega, Wind(c, p) = 0$
This useful lemma says a cycle is a boundary if and only if winds around every point not in its image zero times. The form of this result is dual to a familiar rule from physics: every conservative vector field is a gradient vector field. And it will also be the key to prove the highly nontrivial theorem of De Rham Isomorphism.

Sample application: Brouwer Fixed Point Theorem

As standard prelude to Brouwer fixed point theorem, we first prove a no retraction Lemma, this is the part where the technique of winding number is used.
To keep the poster compact, we include only a heuristic proof.
Lemma: There exists to function $r : D^2 \to S^1$, s.t., $r(z) = z, \forall z \in S^1$.
Proof: Suppose for contradiction such a function exists. Then this r serves as a way to continuously deform the unit speed standard loop (the restriction of r to S^1) and a constant. But a constant has winding number 0, the unit speed standard loop has winding number 1, so we have a contradiction.

Now we are ready to prove the theorem: Let $D^2 = \{z||z| \leq 1\}$. f is any continuous map on D^2, $f : D^2 \to D^2$. Then there exists $z \in D^2$, s.t., $f(z) = z$
Proof: Suppose for the purpose of contradiction that there is such f continuous on D^2, but satisfies $\forall z \in D^2$, $f(z) \neq z$. Then we define a map $r : D^2 \to S^1$ by the following process: Given any input point p, consider the line segment between p and $f(p)$, it is nontrivial because f has no fixed point. Consider the ray emitting from p and points in the direction of the vector with beginning point $f(p)$, end point p. When p is strictly in the interior of S^1, since this ray originates from a point inside the unit circle, it will necessarily intersects the unit circle at a unique point q. When p is on the unit circle, a ray from p may intersect S^1 at two points it points inwards, but here by definition, the ray has direction of a vector that ends at the boundary p, this ray always points outward, hence it intersects S^1 still only at one point q, which is just p itself. See below figure for illustration. Finally $r(p) := q$.

Topics Treated

1. Winding Number as the increment of the angle
2. Winding Number and the zeros of polynomials, Roche's theorem
3. Contour Integral and Cauchy Integral Formula
4. Relation between Cauchy-Goursat Theorem and Green's theorem
5. Winding number and the first fundamental group
6. Winding number and the first De Rham Homology
7. Classical applications of winding numebrs in algebraic topology

Key Definitions

Continuous Definition of Winding Number: Let $\phi : [a, b] \to R^2 - \{0\}$ be a smooth curve, and $\phi(0) = \phi(1)$.The winding number of ϕ around 0 is the Riemann integral:

$$Wind(\phi, 0) = \frac{1}{2\pi} \int_a^b \phi_1'(t) \frac{-\phi_2(t)}{\phi_1(t)^2 + \phi_2(t)^2} + \phi_2'(t) \frac{\phi_1(t)}{\phi_1(t)^2 + \phi_2(t)^2} dt$$

An discrete algorithem for computing winding numbers: Starting with a given function, $f : S^1 \to R^2 - 0$, we use a standard Stone-Weierstrass algorithm to obtain a rational approximation of f. Then we count the zeros and poles of this rational function, but only those lying in the unit circle, the subtraction of the poles from the zeros, counting multiplicities, is the winding number of f.

Chains and cycles:
$\phi : [0, 1] \to \Omega$ is a piece-wise smooth mapping, i.e., there exists a finite partition of the domain such that on each sub-interval, ϕ is smooth, the collection of all such ϕ form a base that generates a free Abelian group, $C_1(\Omega)$, we call it the first chain group. Accordingly, elements of the group, i.e., formal sums of the piece-wise smooth mappings, are called 1-chains.
Consider the free Abelian group generated by Ω as a point set, denoted by $C_0(\Omega)$.
A homomorphism $\partial : C_1(\Omega) \to C_0(\Omega)$ is defined by sending a basis element ϕ to $\phi(1) - \phi(0)$, then extending linearly to the rest of the group. The elements of the kernel of this homomorphism are called cycles.